OCR
RECOGNISING ACHIEVEMENT

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS 2611

Mechanics 5

Tuesday **28 JUNE 2005** Afternoon 1 hour 20 minutes

Additional materials:
Answer booklet
Graph paper
MEI Examination Formulae and Tables (MF12)

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.
- Answer any **three** questions.
- You are permitted to use a graphical calculator in this paper.

INFORMATION FOR CANDIDATES

- The allocation of marks is given in brackets [ ] at the end of each question or part question.
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is being used.
- Final answers should be given to a degree of accuracy appropriate to the context.
- Take $g = 9.8 \text{ m s}^{-2}$ unless otherwise instructed.
- The total number of marks for this paper is 60.

This question paper consists of 4 printed pages.

1 A particle P of mass m, moving in the x-y plane, is subject to a force

$$-ma\omega^2(\sin\omega t\mathbf{i} + 4\cos 2\omega t\mathbf{j}),$$

where t is time and a and ω are positive constants. Initially the particle is at the point $a\mathbf{j}$ and has velocity $a\omega\mathbf{i}$.

(i) Find the velocity of P at time t and show that the position vector of P at time t is

$$\mathbf{r} = a\sin\omega t\mathbf{i} + a\cos 2\omega t\mathbf{j}.$$ [7]

(ii) Find the cartesian equation of the path of P and sketch the path for $t \geqslant 0$. [5]

(iii) Find the power of the force at time t. [2]

(iv) Hence find by integration the work done in the interval $0 \leqslant t \leqslant \frac{\pi}{2\omega}$. Verify that this is equal to the change in kinetic energy of P. [6]

2 **(a)** Two small boats A and B set out from harbours H_A and H_B at the same instant. H_B is 10 km due east of H_A. Each boat travels with constant velocity. Boat A travels at $2\,\mathrm{km\,h^{-1}}$ on a bearing of 045° and B travels at $4\,\mathrm{km\,h^{-1}}$.

(i) Given that B intercepts A, find the bearing on which B travels. [4]

(ii) Given instead that B travels on a bearing of 300°, calculate the magnitude and bearing of the velocity of B relative to A. Hence determine the least distance between the two boats. [7]

(b) Aeroplane X is travelling with velocity $(100\mathbf{i} + 200\mathbf{j})\,\mathrm{m\,s^{-1}}$ and aeroplane Y is travelling with velocity $(50\mathbf{i} - 100\mathbf{j} - 10\mathbf{k})\,\mathrm{m\,s^{-1}}$, where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors in the directions east, north and vertically upwards respectively. Initially the displacement of Y relative to X is $(100\mathbf{i} + 600\mathbf{j} + 100\mathbf{k})\,\mathrm{m}$.

Show that the closest approach of the aeroplanes occurs just after Y passes vertically above X, and find (correct to four significant figures) the least distance between them. [9]

3 A comet of mass m moves around a fixed star at the origin O under the action of a force $-\frac{km\mathbf{r}}{r^3}$, where $\mathbf{r}$ is the position vector of the comet and k is a positive constant.

(i) Show that $r^2\dot{\theta}$ is constant. Denoting this constant by h, show that $\ddot{r} - \frac{h^2}{r^3} = -\frac{k}{r^2}$. [6]

(ii) Given that $r = \frac{1}{u}$, show that $\ddot{r} = -h^2u^2\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}$. Hence find a differential equation relating u and θ. [7]

(iii) The minimum distance between the comet and star is r_0 and occurs at the point $\theta = 0$ when the comet is travelling at speed v_0. Solve the differential equation and hence find r in terms of θ, k, r_0 and v_0. [7]

Question 4 is on the next page.

4 A uniform semicircular lamina L_1 has mass m and radius a. The diameter is AB and the mid-point of AB is O.

(i) Show, by integration, that the moment of inertia of L_1 about an axis through O and perpendicular to L_1 is $\frac{1}{2}ma^2$. [4]

A lamina L_2 is made by attaching a uniform square lamina ABCD of mass λm and side $2a$ to L_1 as shown in Fig. 4. The mid-point of CD is X.

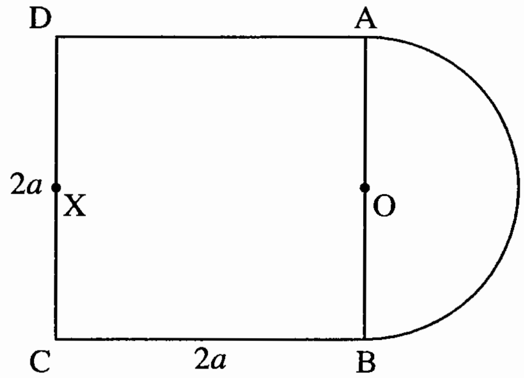


Fig. 4

(ii) Show that the moment of inertia of L_2 about an axis through O and perpendicular to the lamina is given by $I_O = \frac{1}{6}ma^2(3 + 10\lambda)$. [3]

The value of λ is chosen so that the centre of mass of L_2 is at O.

(iii) Find λ and hence show that $I_O = kMa^2$ where M is the mass of L_2 and

$$k = \frac{9\pi + 40}{18\pi + 24}.$$ [4]

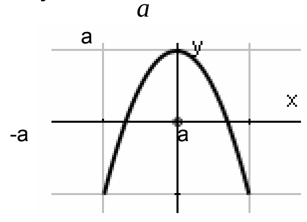
(iv) Find the moment of inertia of L_2 about an axis through X and perpendicular to L_2, in terms of k, M and a. [2]

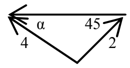
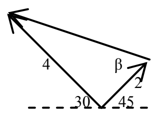
The lamina is suspended from a horizontal axis through X and can rotate freely in a vertical plane. The angle that XO makes with the downward vertical at time t is θ. The lamina is released from rest with $\theta = \alpha$, where $0 < \alpha < \pi$.

(v) Find $\dot{\theta}^2$ in terms of k, g, a, θ and α. [3]

(vi) Show that, if α is sufficiently small, the resulting motion is approximately simple harmonic. [4]

Mark Scheme 2611
June 2005

1(i)	$m\ddot{\mathbf{r}} = -ma\omega^2\left(\sin\omega t\mathbf{i} + 4\cos 2\omega t\mathbf{j}\right)$	M1	use of N2L
	$\dot{\mathbf{r}} = -a\omega\left(-\cos\omega t\mathbf{i} + 2\sin 2\omega t\mathbf{j}\right) + \mathbf{c}_1$	M1 A1	integrating
	$t = 0, \dot{\mathbf{r}} = a\omega\mathbf{i} \Rightarrow \mathbf{c}_1 = \mathbf{0}$	M1	condition
	$\dot{\mathbf{r}} = a\omega\left(\cos\omega t\mathbf{i} - 2\sin 2\omega t\mathbf{j}\right)$	A1	
	$\mathbf{r} = a\left(\sin\omega t\mathbf{i} + \cos 2\omega t\mathbf{j}\right) + \mathbf{c}_2$	M1	integrating
	$t = 0, \mathbf{r} = a\mathbf{j} \Rightarrow \mathbf{c}_2 = \mathbf{0} \Rightarrow \mathbf{r} = a\left(\sin\omega t\mathbf{i} + \cos 2\omega t\mathbf{j}\right)$	E1	
			7
(ii)	$x = a\sin\omega t, y = a\cos 2\omega t$	M1	components and eliminate t
	$y = a(1 - 2\sin^2\omega t)$	B1	trig identity
	$y = a - \dfrac{2x^2}{a}$	A1	
		B1 B1	parabola $(\cap)$ through $(0,a)$ endpoints indicated
			5
(iii)	$P = \mathbf{F}\cdot\mathbf{v} = (-ma\omega^2)(-a\omega)\begin{pmatrix}\sin\omega t\\ 4\cos 2\omega t\end{pmatrix}\cdot\begin{pmatrix}-\cos\omega t\\ 2\sin 2\omega t\end{pmatrix}$	M1	
	$= ma^2\omega^3(8\sin 2\omega t\cos 2\omega t - \sin\omega t\cos\omega t)$	A1	
			2
(iv)	$\text{WD} = \int_0^{\pi/2\omega} P\mathrm{d}t = ma^2\omega^3\int_0^{\pi/2\omega}(4\sin 4\omega t - \frac{1}{2}\sin 2\omega t)\mathrm{d}t$	M1	attempt integral of P
	$= ma^2\omega^3\left[-\frac{1}{\omega}\cos 4\omega t + \frac{1}{4\omega}\cos 2\omega t\right]_0^{\pi/2\omega}$	M1 A1	trig identity and integrate [ ] or constant multiple
	$= ma^2\omega^2(-\frac{1}{4} - \frac{1}{4}) = -\frac{1}{2}ma^2\omega^2$	A1	
	$\Delta\text{KE} = \frac{1}{2}m\left(\lvert\mathbf{0}\rvert^2 - \lvert a\omega\mathbf{i}\rvert^2\right)$	M1	
	$= -\frac{1}{2}ma^2\omega^2$	E1	both results
			6

2(a)(i)	${}_B\mathbf{v}_A$ must be due West		B1	may be implied
		or ${}_B\mathbf{v}_A = \begin{pmatrix} -4\cos\alpha - \sqrt{2} \\ 4\sin\alpha - \sqrt{2} \end{pmatrix}$	M1	diagram or relative velocity vector
	$\dfrac{2}{\sin\alpha} = \dfrac{4}{\sin 45°}$	or $4\sin\alpha - \sqrt{2} = 0$	M1	
	$\alpha = 20.7°$ hence bearing 290.7°		A1	
				4
(ii)		or ${}_B\mathbf{v}_A = \begin{pmatrix} -2\sqrt{3} - \sqrt{2} \\ 2 - \sqrt{2} \end{pmatrix}$	M1	diagram or relative velocity vector
	$\left\|{}_B\mathbf{v}_A\right\|^2 = 4^2 + 2^2 - 2\cdot 4\cdot 2\cos 105°$	or $\left\|{}_B\mathbf{v}_A\right\|^2 = \left(-2\sqrt{3} - \sqrt{2}\right)^2 + \left(2 - \sqrt{2}\right)^2$	M1	calculate magnitude
	$\left\|{}_B\mathbf{v}_A\right\| = 4.913 \text{ km h}^{-1}$		A1	
	$\dfrac{4}{\sin\beta} = \dfrac{4.913\ldots}{\sin 105°}$	or $\tan^{-1}\left(\dfrac{-2\sqrt{3} - \sqrt{2}}{2 - \sqrt{2}}\right)$	M1	calculate angle
	$\beta = 51.8\ldots°$ hence bearing 276.8°		A1	
	$d = 10\sin(\beta - 45°)$		M1	complete method
	$= 1.19$ km		A1	
				7
(b)	${}_Y\mathbf{v}_X = \begin{pmatrix} 20 \\ -100 \\ -10 \end{pmatrix} - \begin{pmatrix} 100 \\ 200 \\ 0 \end{pmatrix} = \begin{pmatrix} -50 \\ -300 \\ -10 \end{pmatrix}$		M1 A1	
	$\overrightarrow{XY} = \begin{pmatrix} 100 \\ 600 \\ 100 \end{pmatrix} - \begin{pmatrix} 50 \\ 300 \\ 10 \end{pmatrix} t$		A1	
	$\left\|\overrightarrow{XY}\right\|^2 = (100 - 50t)^2 + (600 - 300t)^2 + (100 - 10t)^2$ $= 200(463t^2 - 1860t + 1900)$		A1	
	min. when $t = \dfrac{930}{463} = 2.0086\ldots$		M1 A1	any valid method for minimum
	vertically above $\Rightarrow \overrightarrow{XY}$ parallel to $\mathbf{k} \Rightarrow t = 2$		M1	
	hence closest at t = 2.0086… i.e. just after vertical		E1	
	least distance = 79.96		A1	
				9

3(i)	$m\ddot{\mathbf{r}} = -\dfrac{km\mathbf{r}}{r^3}$		
	$m\cdot\dfrac{1}{r}\dfrac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right) = 0$	M1	
	$\Rightarrow r^2\dot{\theta} =$ constant	E1	
	$m(\ddot{r} - r\dot{\theta}^2) = -\dfrac{kmr}{r^3}$	M1 A1	
	$\dot{\theta} = \dfrac{h}{r^2} \Rightarrow \ddot{r} - r\left(\dfrac{h}{r^2}\right)^2 = -\dfrac{k}{r^2}$	M1	
	$\Rightarrow \ddot{r} - \dfrac{h}{r^3}^2 = -\dfrac{k}{r^2}$	E1	
			6
(ii)	$\dot{r} = -\dfrac{1}{u^2}\dot{u} = -\dfrac{1}{u^2}\dfrac{\mathrm{d}u}{\mathrm{d}\theta}\dot{\theta}$	M1	
	$= -\dfrac{1}{u^2}\dfrac{\mathrm{d}u}{\mathrm{d}\theta}hu^2$	M1	
	$= -h\dfrac{\mathrm{d}u}{\mathrm{d}\theta}$	A1	
	$\ddot{r} = -h\dfrac{\mathrm{d}^2u}{\mathrm{d}\theta^2}\dot{\theta} = -h\dfrac{\mathrm{d}^2u}{\mathrm{d}\theta^2}hu^2$	M1	
	$= -h^2u^2\dfrac{\mathrm{d}^2u}{\mathrm{d}\theta^2}$	A1	
	$-h^2u^2\dfrac{\mathrm{d}^2u}{\mathrm{d}\theta^2} - h^2u^3 = -ku^2$	M1	
	$\dfrac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = \dfrac{k}{h^2}$	A1	
			7
(iii)	CF $u = A\sin\theta + B\cos\theta$	M1	
	PI $u = \dfrac{k}{h^2}$	B1	
	$u = \dfrac{k}{h^2} + A\sin\theta + B\cos\theta$	A1	
	r min. $\Rightarrow u$ max. at $\theta = 0 \Rightarrow A = 0$	B1	
	$\theta = 0, r = r_0 \Rightarrow \dfrac{1}{r_0} = \dfrac{k}{h^2} + B \Rightarrow B = \dfrac{1}{r_0} - \dfrac{k}{h^2}$	M1	
	at $\theta = 0$, rad. vel. = 0 $\Rightarrow$ trans. vel., $r\dot{\theta} = v_0 \Rightarrow v_0 = \dfrac{h}{r_0}$	M1 relating h to r_0, v_0	
	$r = \dfrac{{v_0}^2{r_0}^2}{k + ({v_0}^2 r_0 - k)\cos\theta}$	A1	
			7

4(i)	Taking a semicircular 'strip' of radius r and width δr			
	area $\approx \pi r \delta r$ so mass $\approx \dfrac{m\pi r}{\frac{1}{2}\pi a^2}\delta r$	M1		
	mom. of inertia $\approx \dfrac{2mr}{a^2}\delta r \times r^2$	M1		
	$I = \dfrac{2m}{a^2}\int_0^a r^3 \mathrm{d}r = \dfrac{2m}{a^2}\left[\frac{1}{4}r^4\right]_0^a$	M1		
	$= \frac{1}{2}ma^2$	E1		
				4
(ii)	$I_{\text{square}} = \frac{1}{3}\lambda m(a^2+a^2) + \lambda ma^2$	M1		
	$I_{\text{O}} = \frac{1}{2}ma^2 + I_{\text{square}}$	B1		
	$I_{\text{O}} = \frac{1}{2}ma^2 + \frac{5}{3}\lambda ma^2 = \frac{1}{6}ma^2(3+10\lambda)$	E1		
				3
(iii)	$\lambda m \cdot a = m \cdot \dfrac{4a}{3\pi} \Rightarrow \lambda = \dfrac{4}{3\pi}$	B1		
	$I_{\text{O}} = \frac{1}{6}\left(\dfrac{M}{1+\lambda}\right)a^2(3+10\lambda)$	M1		
	$= \dfrac{3+10\lambda}{6(1+\lambda)}Ma^2$	A1		
	$k = \dfrac{3+\frac{40}{3\pi}}{6(1+\frac{4}{3\pi})} = \dfrac{9\pi+40}{18\pi+24}$	E1		
				4
(iv)	$I_{\text{X}} = kMa^2 + M(2a)^2$	M1	parallel axis theorem	
	$= (k+4)Ma^2$	A1		
				2
(v)	$\frac{1}{2}I_{\text{X}}\dot{\theta}^2 - Mg\cdot 2a\cos\theta = -Mg\cdot 2a\cos\alpha$	M1 A1	energy	
	$\Rightarrow \dot{\theta}^2 = \dfrac{4g}{a(k+4)}(\cos\theta - \cos\alpha)$	A1	aef	
				3
(vi)	$2\dot{\theta}\ddot{\theta} = \dfrac{4g}{a(k+4)}(-\sin\theta\,\dot{\theta})$	M1 A1	differentiate or use equation of rotation	
	θ small $\Rightarrow \sin\theta \approx \theta$	M1	approximation	
	$\ddot{\theta} \approx -\dfrac{2g}{a(k+4)}\theta$ i.e. SHM	A1		
				4

2611: Mechanics 5

General Comments

The standard of work varied widely, but most candidates were able to demonstrate some understanding of the principles involved in this unit.

Comments on Individual Questions

1) (i) Most candidates integrated successfully, but a surprising number omitted the arbitrary constants. Although they are zero in this case, they must not be simply ignored, particularly when the answer is given.

(ii) The path was often found successfully, but many of the sketches were incomplete.

(iii) Some were unable to make any progress here, but others knew the relevant formula and were able to apply it successfully.

(iv) Many candidates knew how to find the work done, but algebraic slips were common.

2) (a)(i) Most candidates attempted this with Cartesian vectors rather than geometric methods. It was usually done correctly.

(ii) Again this was usually done correctly.

(b) Candidates generally knew how to find the closest approach, but many did not show that it occurred just after Y passes vertically above X as requested.

3) (i) This standard result was usually done well.

(ii) This was also often done well, but some struggled to get the correct expression for the second derivative.

(iii) Most were able to get the general solution for the differential equation. Most candidates did not know how to use the given condition to find the position of the initial line and just assumed its position. Many candidates were not able to correctly substitute for h in their expression.

4) (i) Most candidates were able to derive the moment of inertia correctly.

(ii) Again, most candidates were able to do this correctly.

(iii) Some candidates were unable to calculate the value of λ.

(iv) Most candidates were able to find the moment of inertia.

(v) Errors in the energy equation were common, so correct answers were rare.

(vi) Few candidates were able to correctly find the angular acceleration, either from the equation of motion or from the energy equation.